

# Interpolation for Hardy Spaces: Marcinkiewicz decomposition, Complex Interpolation and Holomorphic Martingales

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## Abstract

The real and complex interpolation spaces for the classical Hardy spaces  $H^1$  and  $H^\infty$  were determined in 1983 by P.W. Jones. Due to the analytic constraints the associated Marcinkiewicz decomposition gives rise to a delicate approximation problem for the  $L^1$  metric. Specifically for  $f \in H^p$  the size of

$$\inf\{\|f - f_1\|_1 : f_1 \in H^\infty, \|f_1\|_\infty \leq \lambda\}$$

needs to be determined for any  $\lambda \in \mathbb{R}_+$ . In the present paper we develop a new set of truncation formulae for obtaining the Marcinkiewicz decomposition of  $(H^1, H^\infty)$ . We revisit the real and complex interpolation theory for Hardy spaces by examining our newly found formulae.

**Keywords:** Hardy spaces, Holomorphic martingales, complex and real interpolation spaces, Marcinkiewicz decomposition

**MSC:** 60G42, 60G46 32A35

## 1 Introduction

Let  $\mathbb{D} \subset \mathbb{C}$  denote the open unit disk, and let  $\mathbb{T}$  denote its boundary. As usual,  $H^p(\mathbb{T})$  denote the Hardy spaces in the disk. We let  $(\Omega, (\mathcal{F}_t), \mathbb{P})$  denote the filtered Wiener space; we will work with the Hardy spaces of holomorphic random variables on Wiener space  $H^p(\Omega)$ , see §3 for details. In this paper we revisit the real and complex interpolation method for the couples  $(H^1(\mathbb{T}), H^\infty(\mathbb{T}))$  and  $(H^1(\Omega), H^\infty(\Omega))$ . We introduce new truncation formulae to produce a Marcinkiewicz decomposition for  $(H^1(\mathbb{T}), H^\infty(\mathbb{T}))$  and exploit those in combination with tools of Stochastic Analysis, such as holomorphic random variables and stopping time decompositions. In order to make this paper accessible for non-specialists we included an extended discussion of the concepts and tools we employed here.

Let  $(V, \|\cdot\|_V)$  be a Banach space and let  $X_0$  and  $X_1$  be linear subspaces of  $V$ . Assume that  $(X_0, \|\cdot\|_0)$  and  $(X_1, \|\cdot\|_1)$  are Banach spaces and that the formal inclusion maps  $X_i \rightarrow V$  define bounded linear operators. We then say that  $(X_0, X_1)$

form a compatible pair of Banach spaces. We recall now the construction of the complex and real interpolation spaces associated to a compatible pair  $(X_0, X_1)$ .

We first define the complex interpolation spaces associated to a compatible pair  $(X_0, X_1)$ . Let  $S = \{\zeta \in \mathbb{C} : 0 < \Re \zeta < 1\}$ , let  $\overline{S} := S \cup S_0 \cup S_1$  denote its closure, where  $S_0 = \{it : t \in \mathbb{R}\}$ , and  $S_1 = \{1 + it : t \in \mathbb{R}\}$ . Define  $\mathcal{F}(X_0, X_1)$  to be the vector space of all functions  $F : \overline{S} \rightarrow X_0 + X_1$ , satisfying

1.  $F$  is bounded and continuous on  $\overline{S}$ ,
2.  $F$  is analytic on  $S$ ,
3.  $F(S_0) \subseteq X_0, F(S_1) \subseteq X_1$  and the respective restrictions  $F : S_0 \rightarrow X_0$ , and  $F : S_1 \rightarrow X_1$  are continuous.

Equipped with the norm  $\|F\|_{\mathcal{F}} = \max\{\sup_{t \in \mathbb{R}} \|F(it)\|_{X_0}, \sup_{t \in \mathbb{R}} \|F(1+it)\|_{X_1}\}$  the space  $\mathcal{F}(X_0, X_1)$  is a Banach space. Let  $0 < \theta < 1$ . The complex interpolation is defined as

$$(X_0, X_1)_{[\theta]} = \{x \in X_0 + X_1 : \exists F \in \mathcal{F}(X_0, X_1), F(\theta) = x\},$$

equipped with the natural quotient norm  $\|x\|_{[\theta]} = \inf\{\|F\|_{\mathcal{F}(X_0, X_1)} : F(\theta) = x\}$ .

For the compatible couple of Banach spaces  $(L^1, L^\infty)$  the complex interpolation spaces are determined by the M. Riesz Theorem, asserting that

$$(L^1, L^\infty)_{[\theta]} = L^p, \quad 1/p = 1 - \theta$$

with equality of norms,  $\|x\|_{[\theta]} = \|x\|_{L^p}$ , for any  $x \in L^1 + L^\infty$ .

Now we turn to defining the family of real interpolation spaces of a compatible pair  $(X_0, X_1)$  of Banach spaces. Recall that for  $x \in X_0 + X_1$  and  $t > 0$ , the  $K$ -functional with respect to  $(X_0, X_1)$  is defined as

$$K(x, t, X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x_0 \in X_0, x_1 \in X_1, x = x_0 + x_1\}.$$

Given  $0 < \theta < 1$  and  $1 \leq q < \infty$  we define  $(X_0, X_1)_{\theta, q}$  to consist of those  $x \in X_0 + X_1$  for which

$$\|x\|_{\theta, q} = \left( \int_0^\infty [t^\theta K(f, t, X_0, X_1)]^q \frac{dt}{t} \right)^{1/q}$$

is finite. The space  $((X_0, X_1)_{\theta, q}, \|\cdot\|_{\theta, q})$  is a Banach space.

The real interpolation spaces of the couple  $(L^1, L^\infty)$  coincide with the Lorentz-spaces. There exist  $c > 0, C < \infty$  such that for any  $x \in L^1 + L^\infty$ ,

$$c\|x\|_{L^{p, q}} \leq \|x\|_{\theta, q} \leq C\|x\|_{L^{p, q}},$$

whenever  $1/p = 1 - \theta$ , and  $1 < q < \infty$ . Hence we have equality of spaces  $(L^1, L^\infty)_{\theta, q} = L^{p, q}$ , where  $1/p = 1 - \theta$ , and  $1 < q < \infty$  with equivalence of norms. By an argument of J. Marcinkiewicz, the identification  $(L^1, L^\infty)_{\theta, q} = L^{p, q}$  can be obtained from the following decomposition of  $L^p$ , where  $1 < p < \infty$ . For  $x \in L^p$  with  $\|x\|_p \leq 1$ , and  $\lambda > 0$  there exist  $x_0 \in L^1$  and  $x_1 \in L^\infty$  such that

$$x = x_0 + x_1, \quad \|x_1\|_\infty \leq \lambda, \quad \text{and} \quad \|x_0\|_1 \leq C_p \lambda^{1-p}.$$

Putting  $E = \{t \in \mathbb{T} : |x| \geq \lambda\}$  and set

$$x_1(t) = \begin{cases} x(t), & t \notin E \\ \lambda, & t \in E \end{cases}$$

gives  $\|x_1\|_\infty \leq \lambda$ . Since with Hölder's inequality  $\int |x - x_1| \leq 2 \int_E |x| \leq 2|E|^{1-1/p}$ , Chebyshev's inequality yields the desired  $L^1$ -approximation

$$\int |x - x_1| \leq C_p \lambda^{1-p}.$$

We next turn to discussing Hardy spaces  $H^p(\mathbb{T}) \subseteq L^p(\mathbb{T})$ . Recall that, if  $f \in L^p(\mathbb{T})$  then  $f \in H^p(\mathbb{T})$  if and only if, the harmonic extension of  $f$  to  $\mathbb{D}$  gives rise to an analytic function in  $\mathbb{D}$ . P. W. Jones [7] determined the real and complex interpolation spaces for the compatible couple of Banach spaces  $(H^1(\mathbb{T}), H^\infty(\mathbb{T}))$  as follows,

$$(H^1(\mathbb{T}), H^\infty(\mathbb{T}))_{\theta, q} = H^{p, q}(\mathbb{T}), \quad (H^1(\mathbb{T}), H^\infty(\mathbb{T}))_{[\theta]} = H^p(\mathbb{T}), \quad (1.1)$$

where  $1/p = 1 - \theta, 1 < p < \infty$ . We refer also to Jones' [8, 9] for a survey of those results, and for extensions thereof.

To identify the real interpolation spaces for the couple  $(H^1(\mathbb{T}), H^\infty(\mathbb{T}))$ , P. W. Jones [7] established the Marcinkiewicz decomposition for  $H^p(\mathbb{T})$ , where  $1 < p < \infty$ : For any  $f \in H^p(\mathbb{T})$  with  $\|f\|_p \leq 1$ , and  $\lambda > 0$  there exist  $f_0 \in H^1$  and  $f_1 \in H^\infty$  such that

$$f = f_0 + f_1, \quad \|f_1\|_\infty \leq \lambda, \quad \text{and} \quad \|f_0\|_1 \leq C_p \lambda^{1-p}. \quad (1.2)$$

(Note that the Marcinkiewicz decomposition for the  $L^p$  spaces described above, would not preserve analyticity.) We refer to the monograph by Bennett and Sharply [2] for an exposition of Jones' approach to the Marcinkiewicz decomposition for Hardy spaces.

We next discuss the complex interpolation spaces of  $(H^1, H^\infty)$ . In [7] Jones proved that for any  $f \in H^p(\mathbb{T})$  there exists  $F_1 \in \mathcal{F}(H^1, H^\infty)$  such that

$$\|F_1\|_{\mathcal{F}(H^1, H^\infty)} \leq C \|f\|_p$$

and  $\|F_1(\theta) - f\|_p \leq (1/2)\|f\|_p$  where  $1/p = 1 - \theta$ . Replacing  $f$  by  $F_2(\theta) - f$  and iterating gives a sequence  $F_n \in \mathcal{F}(H^1, H^\infty)$  satisfying,

$$\|F_n\|_{\mathcal{F}(H^1, H^\infty)} \leq C 2^{-n} \|f\|_p \quad \text{and} \quad \|F_{n+1}(\theta) - F_n(\theta)\|_p \leq (1/2)^{n+1} \|f\|_p.$$

Jones then puts  $G = \lim_{m \rightarrow \infty} G_m$  where  $G_m = \sum_{n=1}^m F_n$ , and thereby obtains  $G \in \mathcal{F}(H^1, H^\infty)$  satisfying  $\|G\|_{\mathcal{F}(H^1, H^\infty)} \leq 2C \|f\|_p$  and the pointwise constraint  $G(\theta) = f$ . In view of the M. Riesz theorem this yields  $(H^1(\mathbb{T}), H^\infty(\mathbb{T}))_{[\theta]} = H^p(\mathbb{T})$ , where  $1/p = 1 - \theta$ .

The course of developement gave rise to several proofs of Jones' interpolation theorems, for instance in the work by G. Pisier [17, 18], Q. Xu [19], S. Kislyakov [11, 12], S. Kislyakov and Q. Xu [13], P.W. Jones and P.F.X. Müller [10], and in [14, 15].

In this paper we add a new angle to the Marcinkiewicz decomposition for  $H^p(\mathbb{T})$ . We exploit the inner-outer factorization in the space  $H^p(\mathbb{T})$  and reduce the approximation problem (1.2) to the special case where  $f \in H^p(\mathbb{T})$  is an outer function. We

write down a truncation formula that is specifically adjusted to the case of outer functions. The resulting integral estimates are reduced to Lemma 2.2. In the first part of the paper we used Kislyakov's approach as our point of reference.

In the second part of the paper we illustrate further the use of our newly found formulae, and revisit the martingale approach to identifying the complex interpolation spaces for the Banach couple  $(H^1(\mathbb{T}), H^\infty(\mathbb{T}))$ . We will work with the Hardy spaces of holomorphic random variables on Wiener space  $H^p(\Omega)$ . Using the truncation formulae introduced in the first part of the paper, we revisit the stopping time decompositions in [15] to prove that  $(H^1(\Omega), H^\infty(\Omega))_{[\theta]} = H^p(\Omega)$  where  $1/p = 1 - \theta$ . Doob's embedding  $M : H^p(\mathbb{T}) \rightarrow H^p(\Omega)$  and projection  $N : H^p(\Omega) \rightarrow H^p(\mathbb{T})$  are the operators by which N. Th. Varopoulos [20] relates Hardy spaces of holomorphic random variables to Hardy spaces of analytic functions: We have  $NMf = f$  for any  $f \in H^p(\mathbb{T})$  and  $\|M\|_p \|N\|_p = 1$  for  $1 \leq p \leq \infty$ . By means of the operators  $M, N$  the identity  $(H^1(\Omega), H^\infty(\Omega))_{[\theta]} = H^p(\Omega)$  yields the complex interpolation spaces in (1.1).

## 2 Real Interpolation Spaces $(H^1(\mathbb{T}), H^\infty(\mathbb{T}))_\theta$

**Theorem 2.1.** *There exists  $C_p$ , such that for an arbitrary  $\lambda > 0$  and  $f \in H^p(\mathbb{T})$ ,  $\|f\|_p = 1$ , one can find a function  $f_1 \in H^\infty(\mathbb{T})$  such that (1.2) holds.*

First, we recall some basic facts on Hardy spaces, see e.g. [6, Chap. I–II]. A function  $f(\zeta)$  meromorphic in  $\mathbb{D}$  is said to be of bounded characteristic if

$$\sup_{0 < r < 1} \left\{ \int_{\mathbb{T}} \log^+ |f(rt)| dm(t) \right\} < \infty,$$

where  $dm$  is the Lebesgue measure on  $\mathbb{T}$ . It can be represented as a ratio of two holomorphic functions bounded in the disk, that is,

$$f(\zeta) = \frac{f_+(\zeta)}{f_-(\zeta)}, \quad \sup_{\zeta \in \mathbb{D}} |f_\pm(\zeta)| \leq 1.$$

Such functions form the so-called Schur class  $\mathcal{S}$  (the unit ball of  $H^\infty(\mathbb{T})$ ), in short  $f_\pm \in \mathcal{S}$ . Functions from  $\mathcal{S}$  are represented in the form

$$f_\pm(\zeta) = \prod \frac{\overline{\zeta_j^\pm}}{|\zeta_j^\pm|} \frac{\zeta_j^\pm - \zeta}{1 - \zeta \overline{\zeta_j^\pm}} \exp \left\{ ic^\pm + \int_{\mathbb{T}} \frac{\zeta + t}{\zeta - t} d\tau^\pm(t) \right\},$$

where  $\zeta_j^\pm \in \mathbb{D}$ ,  $c^\pm \in \mathbb{R}$  and  $d\tau^\pm(t)$  are positive measures on  $\mathbb{T}$ . One can decompose  $d\tau^\pm(t)$  into the absolutely continuous  $d\tau_{a.c.}^\pm(t)$  and singular  $d\tau_s^\pm(t)$  part,  $d\tau^\pm(t) = d\tau_{a.c.}^\pm(t) + d\tau_s^\pm(t)$ . The factor

$$f_\pm^{out}(\zeta) = \exp \left\{ ic^\pm + \int_{\mathbb{T}} \frac{\zeta + t}{\zeta - t} d\tau_{a.c.}^\pm(t) \right\}$$

is called the *outer* part of the function  $f_\pm(\zeta)$ . It is defined uniquely (up to a unimodular constant) via boundary values of the modulus of the given function,  $d\tau_{a.c.}^\pm(t) = -\log |f_\pm(t)| dm(t)$ . The remaining part of the function is called the *inner* part  $f_\pm^{in}(\zeta)$ , so that  $f_\pm(\zeta) = f_\pm^{in}(\zeta) f_\pm^{out}(\zeta)$ . The inner part contains the Blaschke

product and the singular component. The function  $f(\zeta)$  is of Smirnov class (or Nevanlinna class  $N^+$ ) if the denominator  $f_-$  is an outer function. Note that any function from  $H^p(\mathbb{T})$  is a function of Smirnov class, and that functions of Smirnov class obey the maximum principle in the following form: if  $f$  is of Smirnov class and its boundary values belong to  $L^p(\mathbb{T})$ , then  $f \in H^p(\mathbb{T})$ .

Our proof of Theorem 2.1 is based on the following lemma, which, we believe, is of an independent interest.

**Lemma 2.2.** *Let  $s$  be of the Schur class and  $s(0) > 0$ . Then*

$$\int |1 - s|^q dm \leq C_q(1 - s(0)), \quad q > 1. \quad (2.1)$$

*Proof.* Since

$$\int |1 - s|^2 dm = \int (1 - s - \bar{s} + |s|^2) dm = 2(1 - s(0)) - \int (1 - |s|^2) dm \quad (2.2)$$

our statement is trivial for  $q \geq 2$ . So, let  $q = 2 - \delta$ ,  $0 < \delta < 1$ .

Since  $\Re(1 - s) \geq 0$  the values of  $(1 - s)^\delta$  belong to the angle

$$\sin \frac{\pi(1 - \delta)}{2} |1 - s|^\delta \leq \Re(1 - s)^\delta.$$

Therefore

$$\begin{aligned} \sin \frac{\pi(1 - \delta)}{2} \int \frac{|1 - s|^2}{|1 - s|^\delta} dm &\leq \Re \int \frac{|1 - s|^2}{(1 - s)^\delta} dm = \Re \int \overline{(1 - s)} (1 - s)^{1 - \delta} dm \\ &= (1 - s(0)) + \Re \int \overline{(1 - s)} ((1 - s)^{1 - \delta} - 1) dm \\ &= (1 - s(0)) + \Re \int (s - 1 + (1 - |s|^2)) \frac{(1 - s)^{1 - \delta} - 1}{s} dm. \end{aligned}$$

Since the function  $F(\zeta) := \frac{(1 - s(\zeta))^{1 - \delta} - 1}{s(\zeta)}$  is bounded in the unit disk  $\mathbb{D}$ , we get

$$\begin{aligned} \sin \frac{\pi(1 - \delta)}{2} \int \frac{|1 - s|^2}{|1 - s|^\delta} dm &\leq (1 - s(0)) \left( 1 - \frac{(1 - s(0))^{1 - \delta} - 1}{s(0)} \right) \\ &\quad + \Re \int (1 - |s|^2) \frac{(1 - s)^{1 - \delta} - 1}{s} dm. \end{aligned}$$

It follows from the integral representation

$$\frac{(1 - u)^{1 - \delta} - 1}{u} = \frac{\sin \pi(1 - \delta)}{\pi} \int_1^\infty \frac{(x - 1)^{1 - \delta}}{x} \frac{dx}{u - x}$$

that  $\Re F(\zeta) \leq 0$ . Therefore

$$\begin{aligned} \sin \frac{\pi(1 - \delta)}{2} \int \frac{|1 - s|^2}{|1 - s|^\delta} dm &\leq (1 - s(0)) \left( 1 + \frac{1 - (1 - s(0))^{1 - \delta}}{s(0)} \right) \\ &\leq 2(1 - s(0)). \end{aligned}$$

That is,

$$\int |1 - s|^q dm \leq \frac{2}{\sin \frac{\pi(q - 1)}{2}} (1 - s(0)) \quad (2.3)$$

for  $1 < q \leq 2$ . □

*Proof of Theorem 2.1.* Let the inner part of  $f_1$  be the inner part of the given function  $f \in H^p$ . We define the outer part of  $f_1$  by its modulus on the boundary

$$|f_1|(t) = \begin{cases} |f(t)|, & t \in \mathbb{T} \setminus E \\ \lambda, & t \in E \end{cases}$$

as before  $E = \{t \in \mathbb{T} : |f| \geq \lambda\}$ . Then

$$f - f_1 = f(1 - s),$$

where  $s$  belongs to the Schur class, moreover

$$s(0) = e^{-\int_E \ln \frac{|f(t)|}{\lambda} dm}. \quad (2.4)$$

Therefore

$$\int |f - f_1| dm \leq \|f\| \left( \int |1 - s|^q dm \right)^{1/q}.$$

We use (2.3)

$$\int |f - f_1| dm \leq C_p (1 - s(0))^{1/q}. \quad (2.5)$$

Since  $1 - e^{-u} \leq u$  and  $\ln u \leq u^p/p$ , from (2.4) and (2.5) we get (1.2).  $\square$

**Remark 2.3.** We point out that in view of reiteration theorems (see Bergh-Löfström [3]) in our case, as for many other interpolation problems, it suffices to apply the counterpart of the estimates of Lemma 2.2 in the trivial case, that is, for the value  $q = 2$ .

### 3 Holomorphic Random Variables

In this section we prepare the tools of Stochastic Analysis we use for identifying the complex interpolation spaces  $(H^1(\Omega), H^\infty(\Omega))_{[\theta]}$  and  $(H^1(\mathbb{T}), H^\infty(\mathbb{T}))_{[\theta]}$ . We base this review of holomorphic random variables on Varopoulos [20], as well as the books by Bass [1] and Durrett [5]. We let  $(\Omega, (\mathcal{F}_t), \mathbb{P})$  denote the filtered Wiener space, and we recall holomorphic martingales on Wiener's filtered probability space  $(\Omega, (\mathcal{F}_t))$ . Those are defined by their Ito integral representations. Let  $(z_t)$  denote complex Brownian motion on  $\Omega$  with normalized covariance process  $\langle z_t, \overline{z_t} \rangle = 2t$  and  $\langle z_t, z_t \rangle = 0$ . Following Varopoulos [20], an integrable  $F : \Omega \rightarrow \mathbb{C}$  is called a holomorphic random variable if there exists a complex valued adapted process  $(X_s)$  so that its Ito integral assumes the form

$$F = F_0 + \int_0^\infty X_s dz_s. \quad (3.1)$$

The subspace of  $L^p(\Omega)$  consisting of holomorphic random variables is denoted  $H^p(\Omega)$ . For a given  $F \in H^p(\Omega)$  and

$$F_t = \mathbb{E}(F | \mathcal{F}_t), \quad (3.2)$$

we call  $(F_t)$  the holomorphic martingale associated to  $F$ . Combining (3.1) and (3.2)

$$F_t = F_0 + \int_0^t X_s dz_s.$$

If  $f \in H^1(\mathbb{T})$  and  $\tau = \inf\{t > 0 : |z_t| > 1\}$  then  $F = f(z_\tau)$  defines a holomorphic random variable. Since  $z_\tau$  is uniformly distributed over  $\mathbb{T}$ , we have  $F \in H^1(\Omega)$  and  $\mathbb{E}|F| = \int |f| dm$ .

Holomorphic random variables are stable under stopping times. Starting with [14] this property of holomorphic random variables was repeatedly used in approximation problems for spaces of analytic functions, see e.g. [10], [16], [15]. Given a  $(\mathcal{F}_t)$  stopping time  $\rho : \Omega \rightarrow \mathbb{R}^+$  and its generated stopping time  $\sigma$  algebra  $\mathcal{F}_\rho$ , then

$$\mathbb{E}(F|\mathcal{F}_\rho) = F_0 + \int_0^\rho X_s dz_s = F_0 + \int_0^\infty 1_{\{s < \rho\}} X_s dz_s. \quad (3.3)$$

As  $\rho : \Omega \rightarrow \mathbb{R}^+$  is a stopping time the process  $(1_{\{s < \rho\}} X_s)$  is adapted to the filtration  $(\mathcal{F}_t)$ , which verifies the above claim that a stopped holomorphic random variable is again holomorphic.

Next, holomorphy is preserved under pointwise multiplication. If  $F, G \in L^2(\Omega)$ , are holomorphic random variables with Ito integrals

$$F = F_0 + \int_0^\infty X_s dz_s, \quad \text{and} \quad G = G_0 + \int_0^\infty Y_s dz_s$$

then the covariance formula yields

$$FG = F_0 G_0 + \int_0^\infty (F_s Y_s + G_s X_s) dz_s. \quad (3.4)$$

Hence  $FG$  is a holomorphic random variable, and the product  $F_t G_t$  is a holomorphic martingale,

$$F_t G_t = \mathbb{E}(FG|\mathcal{F}_t).$$

Finally we remark that holomorphy is preserved under composition with entire functions. If  $f : \mathbb{C} \rightarrow \mathbb{C}$  is analytic and  $F$  is a holomorphic random variable such that the composition  $f(F)$  is integrable. Ito's formula gives

$$f(F) = f(F_0) + \int_0^\infty \partial f(F_s) dz_s. \quad (3.5)$$

Hence  $f(F)$  is a holomorphic random variable and  $f(F_t)$  a holomorphic martingale satisfying

$$f(F_t) = \mathbb{E}(f(F)|\mathcal{F}_t).$$

Summing up, holomorphic random variables are stable under the following operations

- Stopping times,
- Pointwise multiplication,
- Composition with entire functions.

### 3.1 The stochastic Hilbert transform and outer functions.

Let  $R = (R_t)$  be a real valued, square integrable martingale on Wiener space with stochastic integral representation  $R_t = R_0 + \int_0^t Y_s dz_s + \overline{Y}_s d\overline{z}_s$ . We define the stochastic Hilbert transform of  $R$  by putting

$$\mathcal{H}R = i \int_0^\infty \overline{Y}_s d\overline{z}_s - Y_s dz_s, \quad (3.6)$$

Note that  $\mathcal{H}R$  is again real valued and  $\mathcal{H}^2 R = R - \mathbb{E}R$ . For martingales of the form  $R = u(z_\tau)$ , where  $u \in L^2(\mathbb{T})$  is real valued, Ito's formula connects the martingale operator  $\mathcal{H}$  to the classical Hilbert transform  $H$  by the identity  $\mathcal{H}R = (Hu)(z_\tau)$ , see [20].

We also refer to [20] for the following

**Theorem 3.1.** *Let  $R \in L^2(\Omega)$  be real valued with  $\mathbb{E}R = 0$ . Then  $\mathbb{E}(R^2) = \mathbb{E}((\mathcal{H}R)^2)$  and  $R + i\mathcal{H}R \in H^2(\Omega)$ .*

In view of the Burkholder-Gundy inequalities, the stochastic Hilbert transform extends to a bounded operator on  $L^p(\Omega)$  for  $1 < p < \infty$ , and  $\|\mathcal{H}R\|_p \leq C_p \|R\|_p$  where  $C_p = Cp^2/(p-1)$ . Hence the orthogonal projection

$$\frac{1}{2}(\text{Id} + i\mathcal{H}) : L^2(\Omega) \rightarrow H^2(\Omega)$$

extends boundedly to a projection on  $L^p(\Omega)$ .

**Theorem 3.2.** *Let  $R \in L^\infty(\Omega)$  be real-valued and  $F = \exp(R + i\mathcal{H}(R))$ . Then  $F \in H^\infty(\Omega)$ , with  $|F| = \exp(R)$  and  $\mathbb{E}F = \exp \mathbb{E}(R)$ .*

For holomorphic martingales Lemma 2.2 reads as follows.

**Theorem 3.3.** *Let  $F \in H^2(\Omega)$  be of the form  $F = \exp(R + i\mathcal{H}R)$ , where  $R$  is real valued. Let  $A = \{|F| > \lambda\}$  where  $\lambda > 0$ . If  $Z = R1_{\Omega \setminus A} + (\ln \lambda)1_A$  and  $G = \exp(Z + i\mathcal{H}Z)$ , then  $G \in H^\infty(\Omega)$  with  $|G| \leq \lambda$  and*

$$\mathbb{E}|F - G| \leq \mathbb{E}|F|^2/\lambda. \quad (3.7)$$

Moreover if  $S = G/F$  then

$$\mathbb{E}|1 - S|^2 = 2(1 - \mathbb{E}(S)) \leq 2\mathbb{E}(1_A \ln(|F|/\lambda)) \leq (2/\lambda)\mathbb{E}(1_A |F|). \quad (3.8)$$

*Proof.* By homogeneity it suffices to consider the case  $\mathbb{E}|F|^2 = 1$ . By construction,  $G \in H^\infty(\Omega)$  and  $|G| \leq \lambda$ . Moreover  $|G| \leq |F|$ . Hence  $(G/F)$  is contained in the unit ball of  $H^\infty(\Omega)$ , such that by Theorem 3.2

$$\mathbb{E}(G/F) = \exp \mathbb{E} \ln(|G/F|). \quad (3.9)$$

Since  $\mathbb{E}(G/F) \in \mathbb{R}$  we find by arithmetic

$$\mathbb{E}|1 - G/F|^2 = 2(1 - \mathbb{E}(G/F)) - \mathbb{E}(1 - |G/F|^2). \quad (3.10)$$

Using (3.9) and unwinding the construction of  $G$  we have

$$\mathbb{E}(G/F) = \frac{\mathbb{E}(G)}{\mathbb{E}(F)} = \exp \mathbb{E}(Z - R).$$

Since  $(Z - R) = 0$  on  $\Omega \setminus A$  and  $(Z - R) = \ln(\lambda) - \ln(|F|)$  on  $A$  we have, the crucial identity,

$$\mathbb{E}(G/F) = \exp -\mathbb{E}(\ln(|F|/\lambda)1_A).$$

The elementary inequalities  $1 - \exp(-t) \leq t$ ,  $\ln a < a$  and  $\ln a < a^2/2$  for  $a \geq 1$ , together with  $\mathbb{E}|F|^2 = 1$ , give,

$$2(1 - \mathbb{E}(G/F)) \leq 2\mathbb{E}(\ln(|F|/\lambda)1_A) \leq \begin{cases} \mathbb{E}(|F|^2 1_A)/\lambda^2; \\ 2\mathbb{E}(|F| 1_A)/\lambda \end{cases} \quad (3.11)$$



This proves (3.8). Finally, we apply the Cauchy-Schwarz inequality to the product  $F - G = F(1 - G/F)$ . Since  $|G/F| \leq 1$  we may use (3.10) and (3.11) to obtain

$$\mathbb{E}|F - G| \leq (\mathbb{E}|F|^2)^{1/2} (2(1 - \mathbb{E}G/F))^{1/2} \leq \mathbb{E}|F|^2/\lambda.$$

□

### 3.2 Stopping Times

**The stopping time decomposition.** Let  $F \in L^2(\Omega)$ ,  $M > 1$ . Put  $\tau_0 = 0$  and  $\tau_{i+1} = \inf\{t > \tau_i : |\mathbb{E}(F|\mathcal{F}_t)| > M^{i+1}\}$ . Then define

$$F_i = \mathbb{E}(F|\mathcal{F}_{\tau_i}), \quad d_i = F_{i+1} - F_i. \quad (3.12)$$

We have  $|d_i| \leq 2M^{(1+i)}$  with  $\text{supp } d_i \subseteq \{\tau_i < \infty\}$ . Moreover,  $\{d_i\}$  is a martingale difference sequence, hence the decomposition  $F = \mathbb{E}F + \sum_{i=1}^{\infty} d_i$  converges unconditionally in  $L^2(\Omega)$  satisfying

$$\sum_{i=1}^{\infty} \mathbb{E}|d_i|^2 = \mathbb{E}|F - \mathbb{E}F|^2. \quad (3.13)$$

If  $F \in H^2(\Omega)$  then  $F_i, d_i \in H^\infty(\Omega)$ . We refer to  $\{d_i\}$  as the stopping-time decomposition of  $F$

**Doob's maximal function.** Let  $F \in L^2(\Omega)$ . The maximal function  $A(F) = \sup_t |\mathbb{E}(F|\mathcal{F}_t)|$  satisfies  $\|A(F)\|_2 \leq C\|F\|_2$ . It is linked to the stopping time decomposition by  $|F_i| \leq A(F)$ , and  $|d_i| \leq 2A(F)$ .

**Outer functions and truncation.** Let  $F \in L^2(\Omega)$  and  $\log A(F) \in L^1(\Omega)$ . Put

$$R_i = \begin{cases} M^i & \text{on } \{A(F) > M^i\}; \\ AF & \text{on } \{A(F) \leq M^i\}. \end{cases}$$

Put  $\Psi_i = \exp(\ln R_i + iH \ln R_i)$ ,  $\Psi = \exp(\ln AF + iH \ln AF)$  and define

$$w_i = \Psi_{i+8}/\Psi, \quad E_i = \{AF > M^i\}. \quad (3.14)$$

We have then  $w_i \in H^\infty(\Omega)$  with  $|w_i| \leq 1$ , and  $|w_i| \leq |w_{i+1}|$ . By Theorem 3.3 we get  $L^2$  inequalities as follows.

$$\mathbb{E}|1 - w_i|^2 \leq 2(1 - \mathbb{E}w_i) \leq \mathbb{E}(1_{E_{i+8}} \ln(A(F)/M^{i+8})) \leq 2\mathbb{E}(1_{E_{i+8}} A(F))M^{-i-8} \quad (3.15)$$

We refer to  $\{w_i\}$  as the truncation family associated to  $A(F)$ .

### 3.3 Basic Estimates

Let  $F \in H^1(\Omega)$ . The stopping-time decomposition  $\{d_i\}$  of  $F$  and  $\{w_i\}$  the truncation family associated to  $A(F)$  satisfy the following three basic estimates.

**Lemma 3.4.** *For any  $F \in H^1(\Omega)$ , its martingale maximal function  $A(F)$ , its truncation family  $\{w_i\}$  and its stopping time decomposition are related through the pointwise estimates as follows*

$$\sum_{i=0}^{\infty} 1_{E_i} |\Psi_i| \leq 2 \cdot M \cdot A(F).$$

*Proof.* Note  $(AF)^{-1} \leq M^{-i}$  on  $E_i$ , and  $|\Psi_i| \leq M^i$  on  $E_i$ , hence

$$\frac{1_{E_i} |\Psi_i|}{A(F)} \leq \sum_{j \geq i} 1_{E_j \setminus E_{j+1}} \frac{M^i}{M^j}.$$

Summing the above estimates over  $i \in \mathbb{N}$  and evaluating the geometric series  $(\sum_{i \leq j} M^i)$  gives

$$\begin{aligned} \sum_{i=0}^{\infty} 1_{E_i} \frac{R_i}{A(F)} &\leq \sum_{i=0}^{\infty} \sum_{j \geq i} 1_{E_j \setminus E_{j+1}} \frac{M^i}{M^j} \\ &\leq \sum_{j=1}^{\infty} M^{-j} 1_{E_j \setminus E_{j+1}} \left( \sum_{i \leq j} M^i \right) \\ &\leq 2M \sum_{j=1}^{\infty} 1_{E_j \setminus E_{j+1}} \leq 2M. \end{aligned}$$

□

**Lemma 3.5.** *Let  $F \in H^1(\Omega)$ . Then*

$$\sum_{i=0}^{\infty} M^i \mathbb{E}(1_{E_{i+8}} A(F)) \leq M^{-7} \mathbb{E}((A(F))^2).$$

*Proof.* First  $\mathbb{E}(1_{E_{i+8}} A(F)) \leq \sum_{j \geq i+8} \mathbb{P}(E_j) M^j$  holds by elementary integral estimates. Multiplying by  $M^i$  and summing over  $i \in \mathbb{N}$  gives,

$$\begin{aligned} \sum_{i=0}^{\infty} M^i \mathbb{E}(1_{E_{i+8}} A(F)) &\leq \sum_{i=0}^{\infty} \sum_{j \geq i+8} \mathbb{P}(E_j) M^{j+i} \\ &\leq \sum_{j \geq 8} \mathbb{P}(E_j) M^j \left( \sum_{i \leq j-8} M^i \right) \\ &\leq M^{-7} \sum_{j \geq 8} \mathbb{P}(E_j) M^{2j} \\ &\leq M^{-6} \mathbb{E}((A(F))^2). \end{aligned}$$

□

**Lemma 3.6.** *For any  $F \in H^1(\Omega)$ , its martingale maximal function  $A(F)$ , its truncation family  $\{w_i\}$  and its stopping time decomposition  $\{d_i\}$  are related through  $L^2$  estimates by*

$$\mathbb{E} \left( \left| \sum_{i=0}^{\infty} d_i (1 - w_i) \right|^2 \right) \leq M^{-6} \mathbb{E}((AF)^2).$$

*Proof.* Put first

$$S_1 = 2 \sum_{i=0}^{i+7} \sum_{j=1} \mathbb{E} d_i(1-w_i) d_j(1-w_j) \quad \text{and} \quad S_2 = 2 \sum_{i=0} \sum_{j=i+8}^{\infty} \mathbb{E} d_i(1-w_i) d_j(1-w_j)$$

By direct expansion gives  $S_1 + S_2 = \mathbb{E}(|\sum_{i=0}^{\infty} d_i(1-w_i)|^2)$ . We next estimate  $S_1$  and  $S_2$  separately.

Beginning with  $S_1$ , recall that  $|d_i| \leq 2M^{i+1}$ , and that  $\mathbb{E}|(1-w_i)|^2 \leq 2M^{-i-8} \mathbb{E}(1_{E_{i+8}} AF)$ . Hence

$$\mathbb{E}|d_i(1-w_i)|^2 \leq 8M^{i-7} \mathbb{E}(1_{E_{i+8}} AF),$$

and in view of the arithmetic geometric mean

$$\mathbb{E}|d_i(1-w_i) d_j(1-w_j)| \leq 2M^{i-7} \mathbb{E}(1_{E_{i+8}} AF) + 2M^{j-7} \mathbb{E}(1_{E_{j+8}} AF).$$

Taking the sum over  $i \leq j \leq i+7$ , and invoking Lemma 3.5 gives

$$S_1 \leq 2 \sum_i \sum_{j=1}^{i+7} \mathbb{E}|d_i(1-w_i) d_j(1-w_j)| \leq 7M^{-7} \mathbb{E}((AF)^2).$$

Next we turn to estimating  $S_2$ . To this end fix  $j \geq i+8$ . Note that  $d_j$  is supported in  $E_j$ , that  $d_i$  is supported in  $E_i$  and that  $E_j \subseteq E_i$ . Hence using  $|d_i| \leq 2M^i$  and  $|1-w_i| \leq 2$  gives

$$\mathbb{E}|d_i(1-w_i) d_j(1-w_j)| \leq 16M^{i+1} M^{j+1} \mathbb{P}(E_j).$$

Taking the sum over  $j \geq i+8$  and invoking Lemma 3.5 we get the following upper bounds for  $S_2$ ,

$$\begin{aligned} S_2 &\leq \sum_i \sum_{j: j \geq i+8} M^{i+1} M^{j+1} \mathbb{P}(E_j) \\ &\leq \sum_{j > 8} \left( \sum_{i: j \geq i+8} M^{i+1} \right) M^{j+1} \mathbb{P}(E_j) \\ &\leq M^{-6} \mathbb{E}((AF)^2). \end{aligned}$$

□

## 4 The Complex Interpolation Space $(H^1(\Omega), H^\infty(\Omega))_{1/2}$

In this section we give a somewhat simplified proof of the result in [15] that identifies the complex interpolation space  $(H^1(\Omega), H^\infty(\Omega))_{1/2}$ .

**Theorem 4.1.**

$$(H^1(\Omega), H^\infty(\Omega))_{1/2} = H^2(\Omega). \quad (4.1)$$

*Proof.* Fix  $F \in H^2(\Omega)$ . Let  $\{d_i\}$  be the stopping-time decomposition of  $F$  defined by (3.12) and let  $\{w_i\}$  the truncation family associated to  $A(F)$  defined in (3.14). Define the following analytic function on the vertical strip  $\overline{S} = \{\zeta \in \mathbb{C} : 0 \leq \Re \zeta \leq 1\}$  with values in  $H^1(\Omega)$

$$G(\omega, \zeta) = \mathbb{E}F + \sum_{i=0}^{\infty} d_i w_i M^{(1-2\zeta)(1+i)}. \quad (4.2)$$

Without loss of generality we assume  $\|F\|_{H^2(\Omega)} = 1$ . As we discussed in the introduction (see also [3], or [4]), in order to prove the identity (4.1) it suffices the following three estimates

1.  $\|G(\cdot, 1/2) - F\|_{H^2(\Omega)} \leq 1/2$ ,
2.  $\sup_{t \in \mathbb{R}} \|G(\cdot, it)\|_{H^1(\Omega)} \leq C$ ,
3.  $\sup_{t \in \mathbb{R}} \|G(\cdot, 1 + it)\|_{H^\infty(\Omega)} \leq C$ ,

where  $C < \infty$  is a constant independent of  $F \in H^2(\Omega)$ . Accordingly the following three lemmata yield Theorem 4.1. We exploit Lemma 3.6 in the proof of Lemma 4.2, and Lemma 3.4 in the proof of Lemma 4.4. By contract in the proof of Lemma 4.3 we exploit just elementary properties of the stopping-time decomposition  $\{d_i\}$  and the truncation family  $\{w_i\}$ .

**Lemma 4.2.**

$$\|G(\cdot, 1/2) - F(\cdot)\|_{H^2(\Omega)} \leq 1/2 \|F(\cdot)\|_{H^2(\Omega)}$$

*Proof.* Since  $F = \mathbb{E}F + \sum_{i=0}^{\infty} d_i$  and since by (4.2),  $G(\omega, 1/2) = \mathbb{E}F + \sum_{i=0}^{\infty} d_i(\omega)w_i(\omega)$ , we have

$$\|G(\cdot, 1/2) - F\|_2^2 = \left\| \sum_{i=0}^{\infty} d_i(1 - w_i) \right\|_2^2.$$

By Lemma 3.6 we get  $\left\| \sum_{i=0}^{\infty} d_i(1 - w_i) \right\|_2^2 \leq M^{-6} \|AF\|_2^2$ . Recall that Doob's maximal theorem asserts that  $\|AF\|_2 \leq 2\|F\|_2$ . For  $M > 1$  large enough this finishes the proof.  $\square$

**Lemma 4.3.** *For each  $t \in \mathbb{R}$ ,*

$$\|G(\cdot, it)\|_{H^1(\Omega)} \leq C \|F(\cdot)\|_{H^2(\Omega)}^2,$$

*where  $C > 0$  is an absolute constant.*

*Proof.* By (4.2) we have  $G(\cdot, it) = \mathbb{E}F + \sum_{j=0}^{\infty} d_j w_j M^{(it(j+1))} M^{(1+j)}$ . Hence triangle inequality gives

$$\|G(\cdot, it)\|_{H^1(\Omega)} \leq \mathbb{E}|F| + \sum_{j=0}^{\infty} \|d_j\|_1 M^{(1+j)}.$$

Since  $\|d_j\|_1 \leq 2\mathbb{P}(E_j)M^{(1+j)}$  and since by elementary distributional estimates we have

$$\sum_{j=0}^{\infty} \mathbb{P}(E_j)M^{(2j)} \leq C_0 M^2 \mathbb{E}((AF)^2),$$

the estimate of Lemma 4.3 holds with  $C = C(M)$ . Here we used again that by Doob's maximal inequality  $\mathbb{E}((AF)^2) \leq 4\mathbb{E}(|F|^2)$ .  $\square$

**Lemma 4.4.** *For each  $t \in \mathbb{R}$ ,*

$$\|G(\cdot, 1 + it)\|_{H^\infty(\Omega)} \leq C,$$

*where  $C > 0$  is an absolute constant.*

*Proof.* By (4.2) we have  $G(\omega, 1 + it) = \mathbb{E}F + \sum_{j=0}^{\infty} d_j w_j M^{(2it(j+1))} M^{-(1+j)}$ . Hence triangle inequality gives

$$|G(\omega, 1 + it)| \leq \mathbb{E}|F| + \sum_{j=0}^{\infty} |d_j w_j| M^{-(1+j)}. \quad (4.3)$$

Since  $|d_j| \leq 2M^{(1+j)}$  with  $\text{supp } d_j \subseteq E_j$ , and since  $|w_j| \leq |\Psi_j|/A(F)$  we have the right hand side of (4.3) bounded by

$$\mathbb{E}|F| + \sum_{j=0}^{\infty} 1_{E_j} \frac{|\Psi_j|}{A(F)}.$$

It remains to invoke the pointwise estimates of Lemma 3.4 to conclude that  $|G(\omega, 1 + it)| \leq C$  with  $C = C(M)$ .  $\square$

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